PHY 204: Elementary Physics II -- Lecture Notes PHY 204: Elementary Physics II (2021)

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33. RLC parallel circuit. Resonant ac circuits

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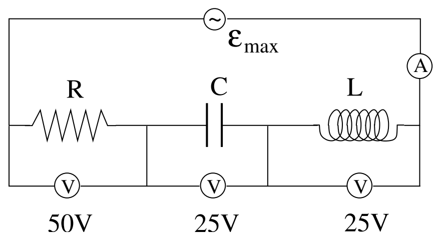
PHY204 Lecture 33 [rln33]

AC Circuit Application (2)

In this RLC circuit, we know the voltage amplitudes V_R, V_C, V_L across each device, the current amplitude $I_{max} = 5\text{A}$, and the angular frequency $\omega = 2\text{rad/s}$.

- Find the device properties R, C, L and the voltage amplitude $\mathcal{E}_{max}$ of the ac source.



tsl305

We pick up the thread from the previous lecture with the quantitative analysis of another RLC series circuit. Here our reasoning must be in reverse direction compared to that on the last page of lecture 32.

Given the current amplitude (measured by the ammeter connected in series) and the voltage amplitudes for each devices (measured by the voltmeters connected in parallel), we can infer the single-device impedances from the ratios:

$$X_R = \frac{50\text{V}}{5\text{A}} = 10\Omega, \quad X_C = \frac{25\text{V}}{5\text{A}} = 5\Omega, \quad X_L = \frac{25\text{V}}{5\text{A}} = 5\Omega.$$

The device properties follow directly:

$$R = 10\Omega, \quad C = 0.1\text{F}, \quad L = 2.5\text{H}.$$

The general expression for the EMF is, we recall from the previous lecture,

$$\mathcal{E}_{max} = \sqrt{V_R^2 + (V_L - V_C)^2}.$$

The minus sign in that expression is due to the fact that the voltages across an inductor and a capacitor connected in series are opposite in phase. Here they are equal in magnitude. Hence we have $\mathcal{E}_{max} = V_R = 50\text{V}$.

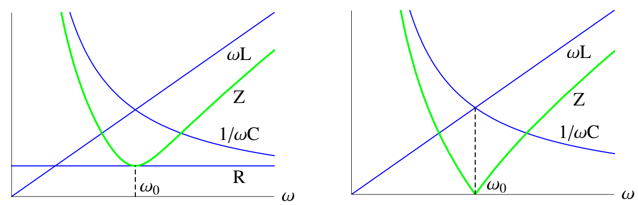
Impedances: RLC in Series (1)

$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

resonance at $\omega_0 = \frac{1}{\sqrt{LC}}$

limit $R \to 0$

$$Z = \left|\omega L - \frac{1}{\omega C}\right|$$



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We have learned that the impedance of the inductor, $X_L = \omega L$, and of the capacitor, $X_C = 1/\omega C$, depend on the angular frequency. The impedance of the resistor is just its resistance, $X_R = R$.

At high frequencies, inductors have large impedance because the (sluggish) current must change rapidly. Capacitors, by contrast, have low impedance at high frequency because they never get a change to charge up significantly.

At low frequencies, the opposite is the case. Inductors become near invisible if the current changes very slowly, whereas capacitors charge up and block any further current until it changes direction.

In an RLC series circuit, the impedance Z is large at low frequencies because of the capacitor and again large at high frequencies because of the inductor. In between, Z has a minimum, realized when the two terms inside the parenthesis are equal in size.

This is the case at the resonance frequency ω_0. At resonance, the (reactive) impedances of the inductor and the capacitor cancel each other. The resulting impedance is the resistance of the resistor.

Removing the resistor from the RLC series circuit means taking the limit $R \to 0$. The resulting expression for the LC series circuit is shown. It touches down to zero at the resonance frequency.

An RLC circuit with very small resistance, when driven at resonance, produces (i) a huge current, which is potentially damaging; (ii) a significant current from a very weak EMF, which is essential for radio receivers.

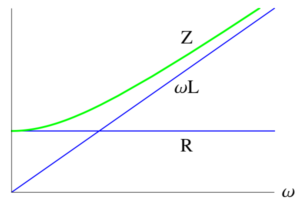
Impedances: RLC in Series (2)

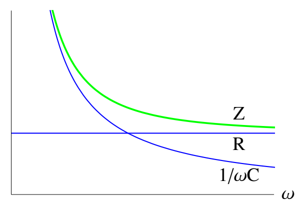
limit $C \to \infty$

$$Z = \sqrt{R^2 + (\omega L)^2}$$

limit $L \to 0$

$$Z = \sqrt{R^2 + \frac{1}{(\omega C)^2}}$$





tsl502

If instead we remove the capacitor from the RLC circuit, we take the limit $C \to \infty$. The resulting expression for the impedance of the RL series circuit is shown and plotted on the left. Conversely, removing the inductor, means, taking the limit $L \to 0$, which yields the results for the RC series circuits shown on the right.

The RL circuit driven by an EMF of given amplitude produces a high-amplitude current at low frequencies and a low-amplitude current at high frequencies. The opposite is the case for the RC circuit.

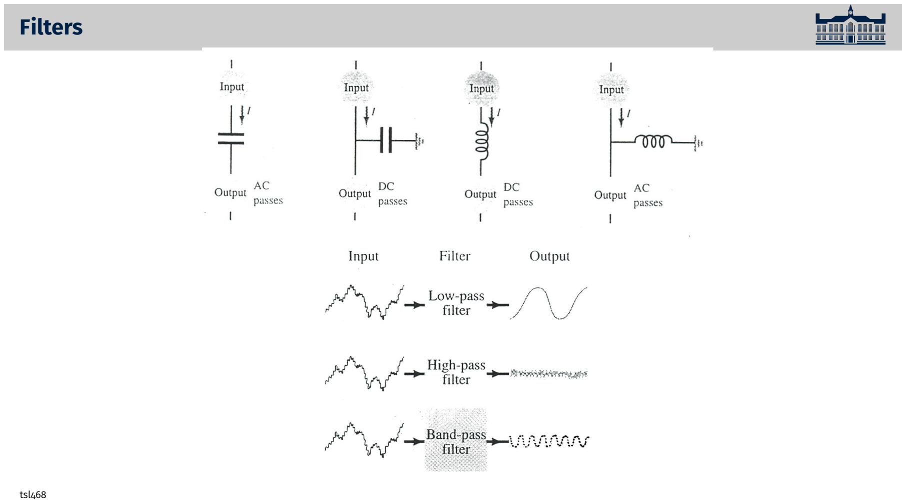
If we have an EMF that is a superposition of oscillations at different frequencies ω_i, it produces a current that is also a superposition of oscillations at the same frequencies but with a different distribution of amplitudes and with different phases:

$$\mathcal{E}(t) = \sum_i \mathcal{E}_{max}^{(i)} \cos(\omega_i t), \quad \Rightarrow \; I(t) = \sum_i \frac{\mathcal{E}_{max}^{(i)}}{Z(\omega_i)} \cos(\omega_i t - \delta_i).$$

Expressions for $Z(\omega_i)$ are shown on this and the previous page. The dependence of δ_i on ω_i was worked out in the previous lecture.

If the circuit is of the RL-type, current amplitudes at high frequencies are being suppressed. If it is of the RC-type, current amplitudes at low frequencies are being suppressed.

These features can be used to filter out noise from electromagnetic signals as illustrated on the next page.



Consider an electromagnetic signal that produces, in an antenna, an EMF in the form a of a superposition of oscillations at different frequencies as noted on the previous page.

We can suppress high-frequency noise by sending the current through an RL series circuit (low-pass filter). Conversely, we can suppress low-frequency noise by sending the same signal through an RC series circuit (high-pass filter).

We can clean up a signal of frequency ω_0 (with a certain bandwidth) by sending it through an RLC series circuit that resonates at ω_0 (band-pass filter). The capacitor will suppress low-frequency noise and the inductor will suppress high-frequency noise.

Note that a low-pass-filter connected to the ground acts as a high-pass filter and vice versa.

RLC Series Resonance (1)

impedance

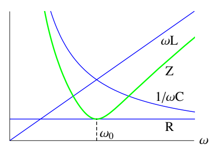
$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

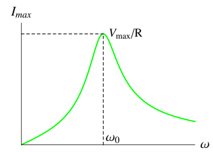
current

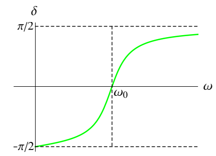
$$I_{max} = \frac{V_{max}}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

phase angle

$$\delta = \frac{\omega L - 1/\omega C}{R}$$







resonance angular frequency:

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

tsl501

Here we take a closer look at the resonance of the RLC series circuit. We assume that the EMF is monochromatic, which means that it oscillates with a single frequency ω: $\mathcal{E}(t) = V_{max}\cos(\omega t)$. We also assume that our ac source is such that we can vary ω and see how the circuit responds.

On this page we focus on the ω-dependence of the amplitude I_{max} and the phase angle δ of the general current expression,

$$I(t) = I_{max}\cos(\omega t - \delta).$$

For completeness, we also show the ω-dependence of the impedance Z.

In the limit $\omega \to 0$, the ac signal turns into a constant voltage such as supplied by a battery. The current amplitude vanishes in that limit. The capacitor blocks a direct current (dc) in a steady state. We recall this fact from our discussion of RC circuits.

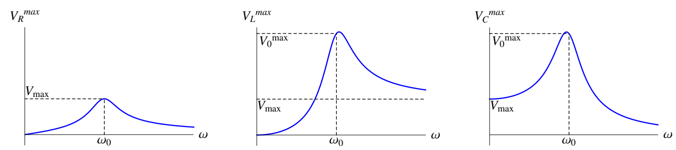
The current amplitude reaches a maximum at resonance ($\omega = \omega_0$). Here the phase angle δ is zero. The current and the EMF are in phase.

At $\omega < \omega_0$, the phase angle is negative. Here the capacitive reactance dominates and suppresses the current amplitude. At $\omega > \omega_0$, the phase angle is positive. Here the inductive reactance dominates and again suppresses the current amplitude.

RLC Series Resonance (2)

resistor | inductor | capacitor

$V_R^{max} = I_{max} R$ $\qquad$ $V_L^{max} = I_{max}\omega L$ $\qquad$ $V_C^{max} = I_{max}/\omega C$



- relaxation times: $\tau_{RC} = RC, \quad \tau_{RL} = L/R$
- angular frequencies: $\omega_L = \dfrac{\omega_0}{\sqrt{1-(\omega_0\,\tau_{RC})^2/2}}, \quad \omega_C = \omega_0\sqrt{1-(\omega_0\,\tau_{RC})^2/2}$
- voltages: $V_0^{max} = V_{max}\,\omega_0\,\tau_{RL}, \quad V_L^{max}(\omega_L) = V_C^{max}(\omega_C) = \dfrac{V_0^{max}}{\sqrt{1-(\omega_0\,\tau_{RC})^2/4}}$

tsl503

On this page, we examine the ω-dependence of the voltage amplitudes across the resistor, the inductor, and the capacitor. The expressions are readily inferred from the expression for $I_{max}(\omega)$ from the previous page.

The slide does not the show the explicit expressions but a graph for each. All three voltages resonate in the sense that their amplitudes are peaked at one particular frequency.

Note that only the voltage across the resistor resonates at ω_0, where the current resonates. The voltage across the inductor resonates ate at ω_L, which is slightly higher and the voltage across the capacitor resonates at ω_C, which is slightly lower.

The expressions for these shifted resonance frequencies and the peak values of all three voltage amplitudes are stated on the slide. In those expressions we have used the relaxation times τ_{RC} and τ_{RL} previously introduced in the contexts of RC circuits and RL circuits, respectively.

Note the distinct behavior of the three voltage amplitudes in the low-frequency and high-frequency limits. The voltage amplitude for the resistor approaches zero in both limits. That is not the case for the other two devices.

The loop rule dictates that the voltage supplied by the ac source is distributed across the three devices. This is true for the instantaneous values, not the amplitudes.

In the limit $\omega \to \infty$, all that voltage is across the inductor and in the limit $\omega \to 0$, all that voltage is across the capacitor for reasons explained earlier.

RLC Parallel Circuit (1)

Applied alternating voltage: $\mathcal{E} = \mathcal{E}_{max} \cos \omega t$

Resulting alternating current: $I = I_{max} \cos(\omega t - \delta)$

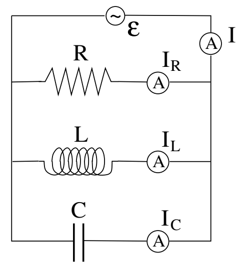
Goals:

- Find I_{max}, δ for given $\mathcal{E}_{max}, \omega$.
- Find currents I_R, I_L, I_C through devices.

Junction rule: $I = I_R + I_L + I_C$

Note:

- All currents are time-dependent.
- In general, each current has a different phase
- I_R has the same phase as $\mathcal{E}$.



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Next we analyze the RLC parallel circuit. The primary goal is the same as for the RLC series circuit, namely to calculate the amplitude I_{max} of the current through the EMF source and its phase angle δ.

In this case, the voltage supplied is the voltage across each device. Each device has a separate current flowing through it, I_R, I_L, and I_C. They are measured by the four ammeters shown, which we again assume to have negligible impedance. Our task is to predict what these currents are and then to assemble them into the current that flows through the EMF source.

The analysis of the RLC series circuit started from the loop rule. Here we start from the junction rule (see slide), keeping in mind that it applies to instantaneous currents, not to amplitudes. Each current has its own phase.

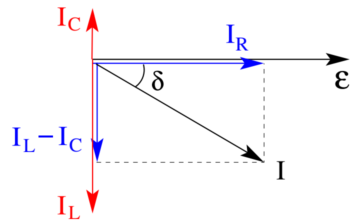
From the single-device circuits analyzed earlier we know the phase relationship between the EMF $\mathcal{E}(t)$ and each of the currents $I_R(t)$, $I_L(t)$, $I_C(t)$: the voltage $\mathcal{E}(t)$ is in phase with $I_R(t)$, lags behind $I_C(t)$ by $\pi/2$ and leads $I_L(t)$ by $\pi/2$ (see pages 2-4 of lecture 32).

We also know the ratios of voltage amplitude and current amplitude for the resistor, the capacitor, and the inductor. These ratios are the single-device impedances. The three current amplitudes inferred from these ratios are listed on the slide of the next page.

From this information it is possible to extract the current amplitude I_{max} and the phase angle δ in the general current expression, $I(t) = I_{max} \cos(\omega t - \delta)$.

RLC Parallel Circuit (2)

Phasor diagram (for $\omega t = \delta$):



Current amplitudes:

- $I_{R,max} = \dfrac{\mathcal{E}_{max}}{X_R} = \dfrac{\mathcal{E}_{max}}{R}$
- $I_{L,max} = \dfrac{\mathcal{E}_{max}}{X_L} = \dfrac{\mathcal{E}_{max}}{\omega L}$
- $I_{C,max} = \dfrac{\mathcal{E}_{max}}{X_C} = \mathcal{E}_{max}\omega C$

Relation between $\mathcal{E}_{max}$ and I_{max} from geometry:

$$\begin{aligned} I_{max}^2 &= I_{R,max}^2 + (I_{L,max} - I_{C,max})^2 \\ &= \mathcal{E}_{max}^2 \left[\frac{1}{R^2} + \left(\frac{1}{\omega L} - \omega C \right)^2 \right] \end{aligned}$$

tsl307

The phasor diagram on the slide helps to explain the extraction of that information. At the instant shown, the EMF $\mathcal{E}$ assumes the maximum value, implying that the current I_R through the resistor, which is in phase with the voltage across it, assumes its maximum value as well. The current I_C through the capacitor leads by $\pi/2$ and the current I_L through the inductor lags by $\pi/2$.

In order that the junction rule is satisfied at all instants in time, we must construct the phasor for the current $I(t)$ geometrically from the phasors of the currents $I_R(t)$, $I_C(t)$ and $I_L(t)$ as carried out in the diagram. This geometric construction accomplishes what we set as our goal:

- It fixes the ratio between the (known) EMF amplitude $\mathcal{E}_{max}$ and the current amplitude I_{max} (to be determined).
- It fixes the phase angle δ between EMF phasor and current phasor.

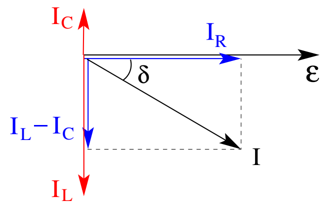
For the first item we use the Pythagorean theorem as worked out at the bottom of the slide. Each quantity on the right-hand side of the first equation contains a factor $\mathcal{E}_{max}$, which we can pull out to arrive at the desired relation. The ratio $\mathcal{E}_{max}/I_{max}$ is the impedance of the RLC parallel circuit (see next page).

RLC Parallel Circuit (3)

Impedance: $\frac{1}{Z} \equiv \frac{I_{max}}{\mathcal{E}_{max}} = \sqrt{\frac{1}{R^2} + \left(\frac{1}{\omega L} - \omega C\right)^2}$

Current amplitude and phase angle:

- $I_{max} = \frac{\mathcal{E}_{max}}{Z} = \mathcal{E}_{max}\sqrt{\frac{1}{R^2} + \left(\frac{1}{\omega L} - \omega C\right)^2}$
- $\tan\delta = \frac{I_{L,max} - I_{C,max}}{I_{R,max}} = \frac{1/\omega L - \omega C}{1/R}$



Currents through devices:

- $I_R = \frac{\mathcal{E}}{R} = \frac{\mathcal{E}_{max}}{R}\cos(\omega t) = I_{R,max}\cos(\omega t)$
- $I_L = \frac{1}{L}\int \mathcal{E}dt = \frac{\mathcal{E}_{max}}{\omega L}\sin(\omega t) = I_{L,max}\cos\left(\omega t - \frac{\pi}{2}\right)$
- $I_C = C\frac{d\mathcal{E}}{dt} = -\omega C\mathcal{E}_{max}\sin(\omega t) = I_{C,max}\cos\left(\omega t + \frac{\pi}{2}\right)$

tsl308

Once we know the impedance Z of the RLC parallel circuit as a function of the circuit specs, ω, R, L, C, we can determine the current amplitude I_{max} from Z and the given $\mathcal{E}_{max}$.

Geometrically, the tangent of the phase angle δ is a ratio of current amplitudes as shown. If we pull out the factor $\mathcal{E}_{max}$ in both numerator and denominator, we are left with an expression that depends on the same specs as the impedance.

Next we calculate the currents across the resistor, the inductor, and the capacitor as functions of time. This is demonstrated on the last three lines of the slide. Here we use the impedances of single devices as established earlier. The last expression on each line reflects the relative orientation of the voltage phasors shown on the slide.

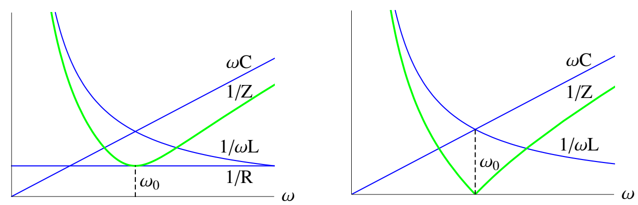
Impedances: RLC in Parallel (1)

$$\frac{1}{Z} = \sqrt{\frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2}$$

resonance at $\omega_0 = \frac{1}{\sqrt{LC}}$

limit $R \to \infty$

$$\frac{1}{Z} = \left|\omega C - \frac{1}{\omega L}\right|$$



tsl504

The familiar and by now well-understood ω-dependence of the reactances X_L and X_C plays out differently in the RLC parallel circuit from what we have found in the RLC series circuit.

In an RLC parallel circuit, the impedance is low when the current is high through one of the three devices. This is the case for the inductor at low ω and for the capacitor at high ω. Note that what is plotted is the inverse impedance.

The resonance frequency ω_0 has the same value as in the series circuit, but it has a different meaning. At resonance, where the two reactances are equal, the currents through the inductor and the capacitor are equal in amplitude but opposite in phase. Hence their net contribution to the current I vanishes. The current I through the ac source is then equal to the current I_R through the resistor.

Removing the resistor form the RLC parallel circuit means taking the limit $R \to \infty$. The resulting expression for inverse impedance of the LC parallel circuit is shown. It touches down to zero at the resonance frequency, implying that the impedance diverges.

An RLC parallel with very small resistance, when driven at resonance, produces huge currents through the inductor and the capacitor. Think of one big current oscillating back and forth within the LC loop and very little current going through the ac source.

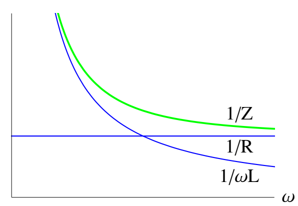
Impedances: RLC in Parallel (2)

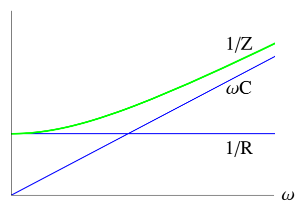
limit $C \to 0$

$$\frac{1}{Z} = \sqrt{\frac{1}{R^2} + \frac{1}{(\omega L)^2}}$$

limit $L \to \infty$

$$\frac{1}{Z} = \sqrt{\frac{1}{R^2} + (\omega C)^2}$$





tsl505

If instead we remove the capacitor from the RLC parallel circuit, we take the limit $C \to 0$. Zero capacitance or infinite resistance (on the previous page) mean, in practice, that no current gets through. These devices are effectively disconnected.

The resulting expression for the impedance of the RL series circuit is shown and plotted on the left. Conversely, removing the inductor, means, taking the limit $L \to 0$, which yields the results for the RC series circuits shown on the right.

In a note of caution we must add that an inductor with infinite inductance would still be invisible to a steady current. But no steady current through an inductor can be established if its inductance is infinite.

We see that the impedance of the RL parallel circuit becomes very small at low frequency and the impedance of the RC parallel circuit very small at high frequency.

Note that ω is, strictly speaking, the angular frequency. The frequency is $f = \omega/2\pi$. The SI unit of the former is rad/s and that of the latter is Hertz [Hz]. It is common, albeit a bit confusing, to say or write *high-frequency* or *low-frequency* and mean, respectively, high or low angular frequency.

RLC Parallel Resonance (1)

impedance

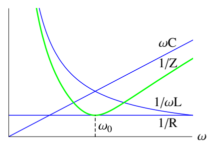
$$\frac{1}{Z} = \sqrt{\frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2}$$

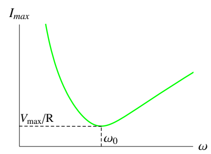
current

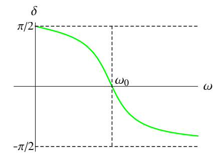
$$I_{max} = \frac{V_{max}}{Z}$$

phase angle

$$\delta = \frac{1/\omega L - \omega C}{1/R}$$







resonance angular frequency:

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

tsl506

Here we take a closer look at the resonance of the RLC parallel circuit, driven by the EMF $\mathcal{E}(t) = V_{max}\cos(\omega t)$ with variable angular frequency ω.

The circuit responds with a current,

$$I(t) = I_{max}\cos(\omega t - \delta),$$

where the amplitude I_{max} and the phase angle δ depend on ω as shown on the slide. For completeness, we also show the ω-dependence of the inverse impedance $1/Z$.

In the limit $\omega \to 0$, the ac signal turns into a constant voltage such as supplied by a battery. The current amplitude diverges in that limit, for which the inductor is responsible. It is invisible to direct currents.

The current amplitude reaches a minimum at resonance ($\omega = \omega_0$). Here the phase angle δ is zero, which should not be a surprise. The current through the resistor produces no phase shift relative to the EMF. The currents through the inductor and the capacitor have the same amplitude and opposite phase. They do not contribute to the current through the ac source.

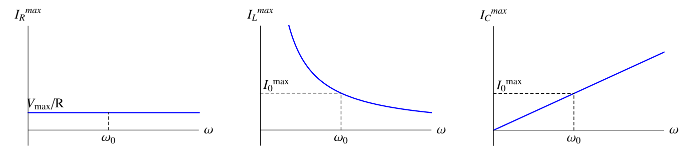
In the high-frequency limit, $\omega \to \infty$, the current amplitude again diverges, but for a different reason. The current through the capacitor is responsible for that effect. The capacitor gets no chance at high frequency to charge up significantly and produce a significant voltage.

At $\omega < \omega_0$, the phase angle is positive. Here the inverse inductive reactance dominates by enhancing the current through the inductor. At $\omega > \omega_0$, the phase angle is negative. Here the inverse capacitive reactance dominates by enhancing the current through the capacitor.

RLC Parallel Resonance (2)

resistor | inductor | capacitor

$I_R^{max} = V_{max}/R$ $\quad$ $I_L^{max} = V_{max}/\omega L$ $\quad$ $I_C^{max} = V_{max}\,\omega C$



currents at resonance:

$$I_R^{max} = \frac{V_{max}}{R}, \quad I_L^{max} = I_C^{max} = I_0^{max} = V_{max}\sqrt{\frac{C}{L}}.$$

tsl507

The enhanced currents mentioned at the bottom of the previous page are evident in the graphs on this slide. It shows the amplitudes of the current through each device. For the resistor that amplitude is independent of ω as is the impedance $X_R = R$.

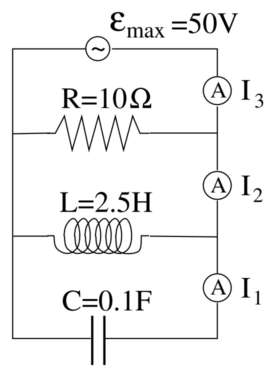
The current through the inductor is enhanced at low ω and the current through the capacitor is enhanced at high ω.

The current amplitudes at resonance are the same for the inductor and the resistor, as mentioned before, and their phases are opposite.

AC Circuit Application (3)

Find the current amplitudes I_1, I_2, I_3

(a) for angular frequency $\omega = 2$rad/s,
(b) for angular frequency $\omega = 4$rad/s.



tsl309

This is the quiz for lecture 33.

We analyze a specific RLC parallel circuit, run at two different angular frequencies. We work out part (b) on this page. The quiz answers part (a).

Part (b): It is useful to first determine the single-device impedances:

$$X_R \doteq R = 10\Omega, \quad X_L \doteq \omega L = 10\Omega, \quad X_C \doteq \frac{1}{\omega C} = 2.5\Omega.$$

The current amplitudes thus become,

$$I_R^{max} = \frac{\mathcal{E}_{max}}{X_R} = 5\text{A}, \quad I_L^{max} = \frac{\mathcal{E}_{max}}{X_L} = 5\text{A},$$

$$V_C^{max} = \frac{\mathcal{E}_{max}}{X_C} = 20\text{A}.$$

We recall the phase relationships between these three currents as encoded in the phasor diagrams for the RLC parallel circuit to conclude that

$$I_1^{max} = I_C^{max} = 20\text{A}, \quad I_2^{max} = |I_L^{max} - I_C^{max}| = 15\text{A},$$

To answer the last question we take a look at the phasor diagram on the previous page and recognize that the voltages across inductor and capacitor are opposite in phase. Hence we have

$$V_{LC}^{max} = |V_L^{max} - V_C^{max}| = 98.6\text{V}.$$

$$I_3^{max} = \sqrt{(I_R^{max})^2 + (I_2^{max})^2} = 15.8\text{A}.$$

Part (a): your turn!